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Continuous limits for the Kac–Van Moerbeke hierarchy and for their restricted flows*

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Abstract. The continuous limit for the Kac–Van Moerbeke (KvM) hierarchy, for their bi-Hamiltonian formulation, recursion relation and square eigenfunction relation is studied. A new family of integrable symplectic maps (ISM) are reduced from the KvM hierarchy via constraint for a higher flow of the hierarchy in terms of square eigenfunctions. Their integrability is deduced from the discrete zero-curvature representation of the KvM hierarchy. It is shown that these ISMs provide maps which approximate many well known integrable mechanical systems (e.g. Neumann, Garnier) embedded into the KdV hierarchy as their restricted flows.

1. Introduction

It is a very remarkable fact that all stationary and restricted flows of the KdV hierarchy are equivalent to finite-dimensional integrable Hamiltonian systems (FDIHS) (see, for example, [1–3]). In practice solutions of such Hamiltonian systems are computed numerically with the use of standard discretization procedures (e.g. the multistep method) which are non-integrable and introduce numerical chaos into the computational process. Much better approximation procedures are exhibited by symplectic maps [4] and even better by integrable symplectic maps which stay on invariant tori of its integrals of motion [5].

A natural and interesting question is to construct integrable discretization for the integrable Hamiltonian systems which follow from restricted flows of soliton hierarchies. By restricted flows of a soliton hierarchy we mean sets of ODEs invariant with respect to the action of all flows of this hierarchy which are constructed in the following way: they consist of a fixed number of copies of the spectral problem and of a restriction for a (higher) flow of the hierarchy in terms of square eigenfunctions. It has been shown [2, 3, 6, 7] that in many instances these restricted flows are finite-dimensional integrable Hamiltonian systems. A general procedure for constructing integrable symplectic maps as restricted flows of a *difference soliton hierarchy of equations, which follow from the proper discretization of the* continuous spectral problem, has been introduced in [8–10]. We suppose that the hierarchy of integrable discrete systems is associated with a discrete isospectral problem and possesses Hamiltonian structure. Then we consider the system consisting of N copies of the spectral problem and of constraint relating the variational derivatives of Hamiltonian functions and

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eigenvalues. This system is also invariant under all flows in the hierarchy, and give rises naturally to discrete Euler–Lagrange equations. In many instances these systems have the form of integrable symplectic maps (ISM). However, it is not a trivial task to show that these maps go to the Hamiltonian systems defined as restricted flows of the continuous hierarchy. In the best case one would like to show that all structures of the discrete hierarchy, such as the spectral problem, recursion relation, and square eigenfunction relation and Hamiltonian structure, together with integrable maps go to the corresponding objects of the continuous hierarchy.

In this paper we study the Kac–Van Moerbeke (KvM) hierarchy, and show how their Hamiltonian structure, recursion relation and square eigenfunction relation converge to those for the KdV hierarchy. We will show that the restricted flows of the KvM hierarchy can be transformed into symplectic maps, and integrals of motion and integrability for these symplectic maps can be deduced directly from the discrete zero-curvature representation for the hierarchy. We find that these ISMs are discrete version of restricted flows of the KdV hierarchy.

Discrete versions of several classical integrable systems are investigated in [11]. To describe such a discrete system a variational principle is taken as a starting point, and the Lax representation for the discrete integrable system is found via a factorization of certain matrix polynomials in [11]. In our approach the starting point for reducing discrete maps is a hierarchy of integrable discrete systems with Lax representation and Hamiltonian structure, and the property of these maps, such as the Lax representation, is directly deduced from one of the hierarchy. It is easy to find that our approach and the approach mentioned above are quite different and used to treat different subjects.

This paper is organized as follows. In the next section, we briefly describe a zero-curvature representation for the KvM hierarchy. Then in section 3 we show how the Hamiltonian structure, the recursion relation and the square eigenfunction relation for the KvM hierarchy converge to those for the KdV hierarchy, and find a sequence of equations in the KvM hierarchy which has the KdV hierarchy as a continuous limit. In section 4, we show that restricted flows of the KvM hierarchy can be transformed into a symplectic maps, and their integrability can be deduced direcly from that of KvM hierarchy by means of the discrete zero-curvature representation. Finally in section 5 we prove that restricted flows of the KvM hierarchy go to restricted flows of the KdV hierarchy.

2. The KvM hierarchy and the KdV hierarchy

2.1. *The KvM hierarchy*

We now briefly present the discrete zero-curvature representation for the KvM hierarchy which can be deduced from that for Toda hierarchy in [12]. Consider the following discrete isospectral problem [13],

$$(E + vE^{(-1)})y = \lambda y \tag{2.1}$$

where $v = v(n,t)$ and $y = y(n,t)$ depend on integers $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, λ is the spectral parameter, shift operator E and difference operator D are defined as

$$(Ef)(n) = f(n+1) \qquad (Df)(n) = (E-1)f(n) \qquad n \in \mathbb{Z}. \tag{2.2}$$

Throughout this paper we write $f^{(k)} = E^{(k)} f$. The scalar spectral problem (2.1) is equivalent to the following matrix spectral problem:

$$E\psi = U\psi \qquad U = U(v, \lambda) = \begin{pmatrix} 0 & 1 \\ -v & \lambda \end{pmatrix} \tag{2.3}$$

with

$$\psi = (\psi_1, \psi_2)^t \equiv (E^{(-1)}y, y)^t . \tag{2.4}$$

To derive the hierarchy of evolution equations associated with (2.3), we first solve the stationary discrete zero-curvature equation [12]

$$(E\Gamma)U - U\Gamma = 0 . \tag{2.5}$$

By substituting

$$\Gamma = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{i=0}^{\infty} \Gamma_i \lambda^{-i} = \sum_{i=0}^{\infty} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i} \tag{2.6}$$

into equation (2.5) and taking the initial value as

$$a_0 = \tfrac{1}{2} \qquad b_0 = 0 \qquad b_1 = -1 \tag{2.7}$$

we find that

$$a_2 = v \qquad a_4 = v(v^{(-1)} + v + v^{(1)}) \qquad b_3 = -v - v^{(-1)} \tag{2.8a}$$

$$c_1 = v \qquad c_3 = v(v + v^{(1)}), \ldots \tag{2.8b}$$

and

$$b_{2i} = c_{2i} = a_{2i+1} = 0 \qquad i = 0, 1, \ldots \tag{2.8c}$$

$$c_{2i+1} = -v b_{2i+1}^{(1)} \qquad b_{2i+1} = -(a_{2i}^{(-1)} + a_{2i}) . \tag{2.8d}$$

Moreover, the quantities

$$P_{2i} \equiv \frac{a_{2i}}{v} \tag{2.9a}$$

satisfy

$$G P_{2i} = J P_{2i+2} \qquad i = 0, 1, \ldots \tag{2.9b}$$

where J and G are Hamiltonian operators defined as

$$\begin{aligned} J &= v\big(E^{(-1)} - E\big)v \\ G &= v\big[vE^{(-1)} + v^{(-1)}E^{(-1)} + v^{(-1)}E^{(-2)} - vE - v^{(1)}E^{(2)} - v^{(1)}E\big]v . \end{aligned} \tag{2.10}$$

The first P_{2i} read

$$P_0 = \frac{1}{2v} \qquad P_2 = 1 \qquad P_4 = v + v^{(-1)} + v^{(1)}$$
$$P_6 = v^{(1)}(v + v^{(1)} + v^{(2)}) + v(v + v^{(-1)} + v^{(1)}) + v^{(-1)}(v + v^{(-1)} + v^{(-2)}) + v^{(-1)}v^{(1)}. \tag{2.11}$$

We set the auxiliary linear problem as

$$\psi_{t_m} = V_{2m}\psi \qquad m = 1, 2, \ldots \tag{2.12}$$

with

$$V_{2m} = (\Gamma\lambda^{2m})_+ + \Delta_{2m}$$
$$= \begin{pmatrix} \sum_{i=0}^{m} a_{2i}\lambda^{2m-2i} & \sum_{i=0}^{m-1} b_{2i+1}\lambda^{2m-2i-1} \\ \sum_{i=0}^{m-1} c_{2i+1}\lambda^{2m-2i-1} & -\sum_{i=0}^{m} a_{2i}\lambda^{2m-2i} \end{pmatrix} + \begin{pmatrix} b_{2m+1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.13}$$

The compatibility condition of (2.3) and (2.12) gives rise to the discrete zero-curvature equations (assuming $\lambda_{t_m} = 0$)

$$U_{t_m} = (EV_{2m})U - UV_{2m} \qquad m = 1, 2, \ldots \tag{2.14}$$

which is the Kac–Van Moerbeke (KvM) hierarchy of equations

$$v_{t_m} = v(a_{2m}^{(-1)} - a_{2m}^{(1)}) = JP_{2m} = J\frac{\delta H_{2m}}{\delta v} \qquad m = 1, 2 \ldots. \tag{2.15}$$

By $\frac{\delta}{\delta v}$ we denote discrete variational derivative defined as

$$\frac{\delta f}{\delta v} = \sum_{k\in\mathbb{Z}} E^{(-k)}\frac{\partial f}{\partial v^{(k)}}.$$

We remind the reader that

$$P_{2m} = \frac{\delta H_{2m}}{\delta v} = \frac{a_{2m}}{v} \qquad H_{2m} = -\frac{b_{2m+1}}{2m}. \tag{2.16}$$

Let us define V in terms of Γ by $\Gamma = VU$. Then it is deduced from (2.5) that

$$D\Gamma = [U, V] \qquad \Gamma^{(1)} = UV \tag{2.17}$$

$$D(a^2 + bc) = \tfrac{1}{2}D(\mathrm{Tr}\Gamma^2) = \tfrac{1}{2}[\mathrm{Tr}(UV)^2 - \mathrm{Tr}(VU)^2] = 0 \tag{2.18}$$

where Tr means trace of a matrix. In the same way as given by [14], we get from (2.12)

$$\Gamma_{t_m} = [V_{2m}, \Gamma] \tag{2.19}$$

which yields

$$2\frac{\mathrm{d}}{\mathrm{d}t_m}(a^2 + bc) = \frac{\mathrm{d}}{\mathrm{d}t_m}\mathrm{Tr}\Gamma^2 = \frac{\mathrm{d}}{\mathrm{d}t_m}\mathrm{Tr}[V_{2m}, \Gamma^2] = 0. \tag{2.20}$$

The adjoint equation of (2.1) is

$$(E^{(-1)} + Ev)y^* = \lambda y^* \tag{2.21}$$

and in matrix form

$$E^{(-1)}\phi = \phi U \qquad \phi = (\phi_1, \phi_2) \equiv (-E(vy^*), y^*). \tag{2.22}$$

The adjoint equation of (2.12) reads

$$E^{(-1)}\phi_{t_m} = -(E^{(-1)}\phi)V_{2m}. \tag{2.23}$$

It can be found by a direct calculation that

$$\frac{\delta\lambda}{\delta v} = -\psi_1\phi_2 \tag{2.24a}$$

$$G\frac{\delta\lambda}{\delta v} = \lambda^2 J\frac{\delta\lambda}{\delta v} \qquad \text{or} \qquad (G - \lambda^2 J)(\psi_1\phi_2) = 0. \tag{2.24b}$$

2.2. The KdV hierarchy

For the KdV hierarchy we accept the follwing conventions. It is associated with the Schrödinger spectral problem of the form [14]

$$(\partial^2 + u - \bar{\lambda})\bar{y} = 0 \tag{2.25}$$

which can be written as

$$\partial\bar{\psi} = \overline{U}\bar{\psi} \qquad \overline{U} = \begin{pmatrix} 0 & 1 \\ \bar{\lambda} - u & 0 \end{pmatrix} \tag{2.26}$$

where $\partial = \frac{\partial}{\partial x}$ and

$$\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)^t = (\bar{y}, \bar{y}_x)^t \,. \tag{2.27}$$

The KdV hierarchy then reads

$$u_{t_m} = B_0 \overline{P}_m \tag{2.28}$$

where all $\overline{P}_k$ are determined from the recursion relation

$$B_0 \overline{P}_{k+1} = B_1 \overline{P}_k \qquad k = 0, 1, \ldots \tag{2.29a}$$

$$B_0 = \partial \qquad B_1 = \tfrac{1}{4}\partial^3 + u\partial + \tfrac{1}{2}u_x \,. \tag{2.29b}$$

The first $\overline{P}_k$ are

$$\overline{P}_0 = 2 \qquad \overline{P}_1 = u \qquad \overline{P}_2 = \tfrac{1}{4}(3u^2 + u_{xx})$$
$$\overline{P}_3 = \tfrac{1}{16}(u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3) \,. \tag{2.30}$$

The adjoint equation of (2.25) is

$$(\partial^2 + u - \bar{\lambda})\bar{y}^* = 0 \tag{2.31}$$

and in the matrix form

$$\partial\bar{\phi} = -\bar{\phi}\overline{U} \qquad \bar{\phi} = (\bar{\phi}_1, \bar{\phi}_2) \equiv (-\bar{y}_x^*, \bar{y}^*) \,. \tag{2.32}$$

It is known that

$$\frac{\delta\bar{\lambda}}{\delta u} = -\bar{\psi}_1\bar{\phi}_2 \tag{2.33a}$$

$$B_1 \frac{\delta\bar{\lambda}}{\delta u} = \bar{\lambda} B_0 \frac{\delta\bar{\lambda}}{\delta u} \qquad \text{or} \qquad (B_1 - \bar{\lambda}B_0)(\bar{\psi}_1\bar{\phi}_2) = 0 \,. \tag{2.33b}$$

3. The continuous limits of the KvM hierarchy

In this section we will show that the Hamiltonian structure, the recursion relation and the square eigenfunction relation for the KvM hierarchy converge to those for the KdV hierarchy, and find a sequence of equations in the KvM hierarchy which has the KdV hierarchy as a continuous limit.

Let us consider the KvM hierarchy on a lattice with a small step h. Define

$$v(n) = 1 + u(x)h^2 \qquad y(n) = \alpha\bar{y}(x) \qquad y^*(n) = \alpha\bar{y}^*(x) \qquad \lambda = 2 + \bar{\lambda}h^2 \tag{3.1a}$$

where α is a constant, and

$$E^{(k)}v = 1 + u(x+kh)h^2 \qquad E^{(k)}y = \alpha\bar{y}(x+kh) \qquad E^{(k)}y^* = \alpha\bar{y}^*(x+kh)\,. \tag{3.1b}$$

It is known that the spectral problem operator has the expansion

$$(E + vE^{(-1)} - \lambda)y = \alpha h^2(\partial^2 + u - \bar{\lambda})\bar{y} + \mathrm{O}(h^3) \tag{3.2a}$$
$$(Ev + E^{(-1)} - \lambda)y^* = \alpha h^2(\partial^2 + u - \bar{\lambda})\bar{y}^* + \mathrm{O}(h^3)\,. \tag{3.2b}$$

Then we find (see appendix A) that

$$G - \lambda^2 J = -8(B_1 - \bar{\lambda}B_0)h^3 + \mathrm{O}(h^5)\,. \tag{3.3}$$

Notice that

$$y = \psi_2 = \alpha\bar{y} = \alpha\bar{\psi}_1 \qquad y^* = \phi_2 = \alpha\bar{y}^* = \alpha\bar{\phi}_2 \tag{3.4a}$$

and therefore

$$\psi_1\phi_2 = \psi_2^{(-1)}\phi_2 = \alpha^2\bar{\psi}_1\bar{\phi}_2 + \mathrm{O}(h)\,. \tag{3.4b}$$

Thus it follows from (3.3) that for square eigenfunctions

$$(G - \lambda^2 J)(\psi_1\phi_2) = -8\alpha^2 h^3(B_1 - \bar{\lambda}B_0)(\bar{\psi}_1\bar{\phi}_2) + \mathrm{O}(h^4) \tag{3.5}$$

which implies that the continuous limit of (2.24*b*) gives (2.33*b*). It is known [14] that for the properly defined square eigenfunctions

$$\psi_1\phi_2 = \sum_{i=0}^{\infty} P_{2i}\lambda^{-2i} \tag{3.6}$$

$$\bar{\psi}_1\bar{\phi}_2 = \sum_{k=0}^{\infty} \overline{P}_k\bar{\lambda}^{-k} \tag{3.7}$$

so that, due to (3.5), the recurrence relation (2.9*b*) corresponds to the recurrence relation (2.29*a*).

Also it is shown in appendix A that we have the following relationship between $\tilde{P}_{2k}$ and P_k:

Proposition 1.

$$\tilde{P}_{2k} \equiv \sum_{i=0}^{k} \beta_{k,i} P_{2i} = \overline{P}_k h^{2k} + O(h^{2k+1}) \tag{3.8a}$$

where

$$\begin{aligned} &\beta_{k,k} = (\tfrac{1}{4})^{k-1} \qquad \beta_{k,0} = \frac{(-1)^k (2k-1)!!}{2^{k-2} k!} \\ &\beta_{k,i} = \tfrac{1}{4}\beta_{k-1,i-1} - \beta_{k-1,i} \qquad i = 1, \ldots, k-1 . \end{aligned} \tag{3.8b}$$

The first $\tilde{P}_{2k}$ read

$$\begin{aligned} &\tilde{P}_0 \equiv 4P_0 \qquad \tilde{P}_2 \equiv P_2 - 2P_0 \qquad \tilde{P}_4 \equiv \tfrac{1}{4}(P_4 - 6P_2 + 6P_0) \\ &\tilde{P}_6 \equiv \tfrac{1}{16}(P_6 - 10P_4 + 30P_2 - 20P_0), \ldots . \end{aligned} \tag{3.9}$$

From equations (3.8*a*) and (3.3*a*), we obtain

$$J\tilde{P}_{2k} = \sum_{i=0}^{k} \beta_{k,i} J P_{2i} = -2B_0 \overline{P}_k h^{2k+1} + O(h^{2k+2}) \tag{3.10}$$

and

$$v_{t_m} + \frac{1}{2h^{2m-1}} J\tilde{P}_{2m} = h^2(u_{t_m} - B_0 \overline{P}_m) + O(h^3) . \tag{3.11}$$

So we have the following proposition.

Proposition 2. The following sequence of equations in the KvM hierarchy.

$$v_{t_m} = -\frac{1}{2h^{2m-1}} J\tilde{P}_{2m} \qquad m = 1, 2, \ldots \tag{3.12}$$

goes to the KdV hierarchy (2.28) in the continuous limit.

For example, for $m = 2$, we find that the equation

$$v_{t_2} = -\frac{1}{8h^3} v(E^{(-1)} - E) v(v + v^{(-1)} + v^{(1)} - 6) \tag{3.13}$$

has the following KdV equation as a continuous limit:

$$u_{t_2} = \tfrac{1}{4}(6uu_x + u_{xxx}) . \tag{3.14}$$

4. New integrable symplectic map

We consider for N distinct λ_j, $j = 1, \ldots, N$, the following system of equations consisting of replicas of (2.3) and (2.22) as well as of the constraint for variational derivatives for conserved quantities H_{2k_0} (for a fixed k_0) and eigenvalue λ_j

$$E\psi_{1j} = \psi_{2j} \qquad E\psi_{2j} = -v\psi_{1j} + \lambda_j\psi_{2j} \qquad j = 1, \ldots, N \tag{4.1a}$$

$$E^{(-1)}\phi_{1j} = -v\phi_{2j} \qquad E^{(-1)}\phi_{2j} = \phi_{1j} + \lambda_j\phi_{2j} \qquad j = 1, \ldots, N \tag{4.1b}$$

$$\frac{\delta H_{2k_0}}{\delta v} - \sum_{j=1}^{N} \frac{\delta\lambda_j}{\delta v} = 0. \tag{4.1c}$$

We shall denote the inner product in $\mathbb{R}^N$ by $\langle\cdot,\cdot\rangle$ and shall use the following notation:

$$\Psi_i = (\psi_{i1}, \ldots, \psi_{iN})^t \qquad \Phi_i = (\phi_{i1}, \ldots, \phi_{iN})^t \qquad i = 1, 2$$
$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N).$$

By substituting (2.24*a*) into (4.1*c*), we get

$$E\Psi_1 = \Psi_2 \qquad E\Psi_2 = -v\Psi_1 + \Lambda\Psi_2 \tag{4.2a}$$

$$E^{(-1)}\Phi_1 = -v\Phi_2 \qquad E^{(-1)}\Phi_2 = \Phi_1 + \Lambda\Phi_2 \tag{4.2b}$$

$$\frac{\delta H_{2k_0}}{\delta v} = -\langle\Psi_1, \Phi_2\rangle. \tag{4.2c}$$

As argued in [8–10], the system equation (4.2) is invariant with respect to the action of all flows of the KvM hierarchy. So (4.2) is expected to give an integrable symplectic map (ISM). We shall show that integrals of motion and integrability of (4.2) can be derived from the discrete zero-curvature representation for KvM hierarchy. Following the procedure in [15], we can introduce canonical coordinates (q, p) for (4.2):

$$q = (q_1, \ldots, q_{N_1})^t \qquad p = (p_1, \ldots, p_{N_1})^t \tag{4.3}$$

and define Poisson bracket for any pair of functions f, g and any (q, p) as follows:

$$\{f, g\}_{q,p} = \sum_{j=1}^{N_1} \left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} \right) \tag{4.4}$$

such that (4.2) can be cast in canonical form of a symplectic map:

$$Eq_i = f_i(q(n), p(n)) \qquad Ep_i = g_i(q(n), p(n)) \qquad i = 1, \ldots, N_1 \tag{4.5}$$

where f_i, g_i satisfy

$$\{f_i, f_j\} = \{g_i, g_j\} = 0 \qquad \{f_i, g_j\} = \delta_{i,j}. \tag{4.6}$$

Now we present the first two symplectic maps obtained from (4.2) as examples.

(i) For $k_0 = 1$, equation (4.2*c*) reads

$$\langle\Psi_1, \Phi_2\rangle = -1 \tag{4.7}$$

which together with (4.2*b*) leads to

$$v = \langle \Psi_1, E^{(-1)}\Phi_1 \rangle = \langle \Psi_1, \widetilde{\Phi}_1 \rangle . \tag{4.8}$$

Throughout this paper, we denote $\widetilde{\Phi}_i = \Phi_i^{(-1)}$, $i = 1, 2$. By substitution of (4.8), we obtain from (4.2*a*) and (4.2*b*) the following map:

$$E\Psi_1 = \Psi_2 \qquad E\Psi_2 = -\langle \Psi_1, \widetilde{\Phi}_1 \rangle \Psi_1 + \Lambda\Psi_2 \tag{4.9a}$$

$$E\widetilde{\Phi}_1 = \frac{1}{\langle \Psi_1, \widetilde{\Phi}_1 \rangle}\Lambda\widetilde{\Phi}_1 + \widetilde{\Phi}_2 \qquad E\widetilde{\Phi}_2 = -\frac{1}{\langle \Psi_1, \widetilde{\Phi}_1 \rangle}\widetilde{\Phi}_1 . \tag{4.9b}$$

For equation (4.9) the canonical coordinates (q, p) are defined as follows:

$$\begin{aligned} q &= (q_1, \ldots, q_{2N})^t \equiv (\psi_{11}, \ldots, \psi_{1N}, \psi_{21}, \ldots, \psi_{2N})^t \qquad N_1 = 2N \\ p &= (p_1, \ldots, p_{2N})^t \equiv (\tilde{\phi}_{11}, \ldots, \tilde{\phi}_{1N}, \tilde{\phi}_{21}, \ldots, \tilde{\phi}_{2N})^t . \end{aligned} \tag{4.10}$$

It is easy to verify that (4.6) for (4.9) holds, so (4.9) defines a symplectic map. From (4.9*b*), we have $\langle \Psi_1, \Phi_2 \rangle = \langle \Psi_1, E\widetilde{\Phi}_2 \langle = -1$, so $\langle \Psi_1, \Phi_2 \rangle = -1$ in (4.7) is not really a constraint for (4.9).

(ii) For $k_0 = 2$, it is found from (4.2*c*) that

$$v + v^{(-1)} + v^{(1)} = -\langle \Psi_1, \Phi_2 \rangle . \tag{4.11}$$

Define

$$q_{2N+1} = v \qquad p_{2N+1} = v^{(-1)} \tag{4.12}$$

then the system (4.2) with (4.2*c*) given by (4.11) can be rewritten in the canonical form

$$E\Psi_1 = \Psi_2 \qquad E\Psi_2 = -q_{2N+1}\Psi_1 + \Lambda\Psi_2 \tag{4.13a}$$

$$Eq_{2N+1} = \frac{1}{q_{2N+1}}\langle \Psi_1, \widetilde{\Phi}_1 \rangle - q_{2N+1} - p_{2N+1} \tag{4.13b}$$

$$E\widetilde{\Phi}_1 = \frac{1}{q_{2N+1}}\Lambda\widetilde{\Phi}_1 + \widetilde{\Phi}_2 \qquad E\widetilde{\Phi}_2 = -\frac{1}{q_{2N+1}}\widetilde{\Phi}_1, \tag{4.13c}$$

$$Ep_{2N+1} = q_{2N+1} . \tag{4.13d}$$

For equation (4.13) the canonical coordinates (q, p) are defined as follows:

$$\begin{aligned} q &= (q_1, \ldots, q_{2N+1})^t \equiv (\psi_{11}, \ldots, \psi_{1N}, \psi_{21}, \ldots, \psi_{2N}, v)^t \qquad N_1 = 2N + 1 \\ p &= (p_1, \ldots, p_{2N+1})^t \equiv (\tilde{\phi}_{11}, \ldots, \tilde{\phi}_{1N}, \tilde{\phi}_{21}, \ldots, \tilde{\phi}_{2N}, v^{(-1)})^t \end{aligned} \tag{4.14}$$

it is easy to verify that (4.6) for (4.13) holds, so (4.13) also defines a symplectic map.

We now use (4.13) ($k_0 = 2$) as an example to illustrate how the integrability of the symplectic map can be deduced from that of the KvM hierarchy (2.15).

Lemma. Under (4.13), let us define

$$\begin{aligned} &\tilde{a}_0 = a_0 = \tfrac{1}{2} \qquad \tilde{a}_2 = a_2 = q_{2N+1} \qquad \tilde{b}_1 = b_1 = -1 \qquad \tilde{b}_3 = b_3 = -q_{2N+1} - p_{2N+1} \\ &\tilde{c}_1 = c_1 = q_{2N+1} \qquad \tilde{c}_3 = c_3 = \langle \Psi_1, \widetilde{\Phi}_1 \rangle - q_{2N+1}p_{2N+1} \end{aligned} \tag{4.15a}$$

$$\tilde{a}_{2i+1} = \tilde{b}_{2i} = \tilde{c}_{2i} = 0 \qquad i = 0, 1, \ldots$$

$$\tilde{a}_{2i} = \tfrac{1}{2}(\langle \Lambda^{2i-4}\Psi_1, \widetilde{\Phi}_1 \rangle - \langle \Lambda^{2i-4}\Psi_2, \widetilde{\Phi}_2 \langle) \qquad i = 2, 3, \ldots \tag{4.15b}$$

$$\tilde{b}_{2i+1} = \langle \Lambda^{2i-3}\Psi_1, \widetilde{\Phi}_2 \rangle \qquad \tilde{c}_{2i+1} = \langle \Lambda^{2i-3}\Psi_2, \widetilde{\Phi}_1 \rangle \qquad i = 2, 3, \ldots \tag{4.15c}$$

$$\widetilde{\Gamma} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & -\tilde{a} \end{pmatrix} = \sum_{i=0}^{\infty} \widetilde{\Gamma}_i \lambda^{-i} = \sum_{i=0}^{\infty} \begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ \tilde{c}_i & -\tilde{a}_i \end{pmatrix} \lambda^{-i} \tag{4.15d}$$

then under (4.13) $\widetilde{\Gamma}$ satisfies (2.7), and

$$D(\tilde{a}^2+\tilde{b}\tilde{c})=0 \tag{4.16}$$

and

$$F_k=\sum_{i=0}^{k}\tilde{a}_{2i}\tilde{a}_{2k-2i}+\sum_{i=0}^{k-1}\tilde{b}_{2i+1}\tilde{c}_{2k-2i-1} \qquad k=0,1,\ldots \tag{4.17}$$

are integrals of motion for (4.13).

Proof. It is easy to verify that under (4.13) $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i$ defined by (4.15) satisfy (2.8), namely under (4.13) $\widetilde{\Gamma}$ satisfies (2.5). According to (2.18), (4.16) holds. Substituting (4.15*d*) into (4.16) gives rise to

$$D\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\tilde{a}_{2i}\tilde{a}_{2k-2i}+\sum_{i=0}^{k-1}\tilde{b}_{2i+1}\tilde{c}_{2k-2i-1}\right)\lambda^{-2k}=0$$

which implies that under (4.13) and (4.15) we have

$$D\left(\sum_{i=0}^{k}\tilde{a}_{2i}\tilde{a}_{2k-2i}+\sum_{i=0}^{k-1}\tilde{b}_{2i+1}\tilde{c}_{2k-2i-1}\right)=0 \qquad k=0,1,\ldots.$$

Thus F_k calculated by means of (4.17) and (4.15) are integrals of motion for (4.13). This completes the proof.

By substituting (4.15) into (4.17), we obtain the integrals of motion for (4.13) as follows:

$$F_0=\tfrac{1}{4} \qquad F_1=0 \qquad F_2=-\tfrac{1}{2}(\langle\Psi_1,\widetilde{\Phi}_1\rangle+\langle\Psi_2,\widetilde{\Phi}_2\langle) \tag{4.18a}$$

$$\begin{aligned}F_3=&\tfrac{1}{2}(\langle\Lambda^2\Psi_1,\widetilde{\Phi}_1\rangle-\langle\Lambda^2\Psi_2,\widetilde{\Phi}_2\rangle)-\langle\Lambda\Psi_2,\widetilde{\Phi}_1\rangle+q_{2N+1}\langle\Lambda\Psi_1,\widetilde{\Phi}_2\rangle\\&-p_{2N+1}\langle\Psi_1,\widetilde{\Phi}_1\rangle-q_{2N+1}\langle\Psi_2,\widetilde{\Phi}_2\rangle+q_{2N+1}^2p_{2N+1}+q_{2N+1}p_{2N+1}^2\end{aligned} \tag{4.18b}$$

$$\begin{aligned}F_k=&\tfrac{1}{2}(\langle\Lambda^{2k-4}\Psi_1,\widetilde{\Phi}_1\rangle-\langle\Lambda^{2k-4}\Psi_2,\widetilde{\Phi}_2\rangle)-\langle\Lambda^{2k-5}\Psi_2,\widetilde{\Phi}_1\rangle\\&+q_{2N+1}(\langle\Lambda^{2k-6}\Psi_1,\widetilde{\Phi}_1\rangle-\langle\Lambda^{2k-6}\Psi_2,\widetilde{\Phi}_2\rangle)+q_{2N+1}\langle\Lambda^{2k-5}\Psi_1,\widetilde{\Phi}_2\rangle\\&-(q_{2N+1}+p_{2N+1})\langle\Lambda^{2k-7}\Psi_2,\widetilde{\Phi}_1\rangle+(\langle\Psi_1,\widetilde{\Phi}_1\rangle-q_{2N+1}p_{2N+1})\langle\Lambda^{2k-7}\Psi_1,\widetilde{\Phi}_2\rangle\\&+\sum_{i=0}^{k-5}[\langle\Lambda^{2i+1}\Psi_1,\widetilde{\Phi}_2\rangle\langle\Lambda^{2k-2i-9}\Psi_2,\widetilde{\Phi}_1\rangle\\&+\tfrac{1}{4}\sum_{i=0}^{k-4}(\langle\Lambda^{2i}\Psi_1,\widetilde{\Phi}_1\rangle-\langle\Lambda^{2i}\Psi_2,\widetilde{\Phi}_2\rangle)(\langle\Lambda^{2k-2i-8}\Psi_1,\widetilde{\Phi}_1\rangle-\langle\Lambda^{2k-2i-8}\Psi_2,\widetilde{\Phi}_2\rangle)\\&k=4,5,\ldots.\end{aligned} \tag{4.18c}$$

By using (2.19) and (2.20), we can show (see appendix B for detail) that

$$\{F_k,F_m\}=0 \qquad k,m=0,1,\ldots \tag{4.19}$$

which means that integrals of motion F_k are in involution with respect to (4.4).

Notice that we assume all λ_j to be distinct to have the Vandermonde determinant of $\lambda_1, \ldots, \lambda_N$ different from zero. For a specific N, it can be verified that

$$\frac{\partial(F_2, F_3, \ldots, F_{2N+2})}{\partial(\tilde{\phi}_{11}, \ldots, \tilde{\phi}_{1N}, \tilde{\phi}_{21}, \ldots, \tilde{\phi}_{2N}, p_{2N+1})} \neq 0 \tag{4.20}$$

so $\mathrm{grad} F_k, k = 2, \ldots, 2N + 2$, are linear independent. Thus we have

Proposition 3. The F_k given by (4.18) are functionally independent integrals of motion in involution for (4.13), and the symplectic map (4.13) is completely integrable in the Liouville sense [15].

Similarly, we obtain the integrals of motion for (4.9) as follows:

$$F_0 = \tfrac{1}{4} \qquad F_1 = -\tfrac{1}{2}(\langle\Psi_1, \widetilde{\Phi}_1\rangle + \langle\Psi_2, \widetilde{\Phi}_2\rangle) \tag{4.21a}$$

$$\begin{aligned} F_k = {} & \tfrac{1}{2}(\langle\Lambda^{2k-2}\Psi_1, \widetilde{\Phi}_1\rangle - \langle\Lambda^{2k-2}\Psi_2, \widetilde{\Phi}_2\rangle) - \langle\Lambda^{2k-3}\Psi_2, \widetilde{\Phi}_1\rangle \\ & + \langle\Psi_1, \widetilde{\Phi}_1\rangle\langle\Lambda^{2k-3}\Psi_1, \widetilde{\Phi}_2\rangle + \sum_{i=0}^{k-3}[\langle\Lambda^{2i+1}\Psi_1, \widetilde{\Phi}_2\rangle\langle\Lambda^{2k-2i-5}\Psi_2, \widetilde{\Phi}_1\rangle \\ & + \sum_{i=0}^{k-2}\tfrac{1}{4}(\langle\Lambda^{2i}\Psi_1, \widetilde{\Phi}_1\rangle - \langle\Lambda^{2i}\Psi_2, \widetilde{\Phi}_2\rangle)(\langle\Lambda^{2k-2i-4}\Psi_1, \widetilde{\Phi}_1\rangle - \langle\Lambda^{2k-2i-4}\Psi_2, \widetilde{\Phi}_2\rangle) \\ & k = 2, 3, \ldots \end{aligned} \tag{4.21b}$$

and conclude that the map (4.9) is an ISM.

Finally, for the system (4.2), we define

$$\tilde{a}_{2i+1} = \tilde{b}_{2i} = \tilde{c}_{2i} = 0 \qquad i = 0, 1, \ldots \tag{4.22a}$$

$$\tilde{a}_{2i} = a_{2i} \qquad \tilde{b}_{2i+1} = b_{2i+1} \qquad \tilde{c}_{2i+1} = c_{2i+1} \qquad i = 0, \ldots, k_0 - 1 \tag{4.22b}$$

$$\tilde{a}_{2i} = \tfrac{1}{2}\big(\langle\Lambda^{2i-2k_0}\Psi_1, \widetilde{\Phi}_1\rangle - \langle\Lambda^{2i-2k_0}\Psi_2, \widetilde{\Phi}_2\rangle\big) \qquad i = k_0, k_0 + 1 \ldots \tag{4.22c}$$

$$\tilde{b}_{2i+1} = \langle\Lambda^{2i-2k_0+1}\Psi_1, \widetilde{\Phi}_2\rangle \qquad \tilde{c}_{2i+1} = \langle\Lambda^{2i-2k_0+1}\Psi_2, \widetilde{\Phi}_1\rangle \qquad i = k_0, k_0 + 1, \ldots . \tag{4.22d}$$

Then under (4.2) $\tilde{a}_i, \tilde{b}_i$ and $\tilde{c}_i$ satisfy (2.8), so integrals of motion F_k for (4.2) can be calculated from (4.17) and (4.22). Integrals of motion for (4.5) can be obtained by expressing the F_k in terms of (q, p). Similarly, we consider the time evolution equations for q and p which can be constructed out from (B.1), and show that the F_k are in involution. So (4.5) is an integrable symplectic map.

5. The continuous limits for restricted flows of the KvM hierarchy

We consider the following restricted flows of the KvM hierarchy:

$$E\Psi_1 = \Psi_2 \qquad E\Psi_2 = -v\Psi_1 + \Lambda\Psi_2 \tag{5.1a}$$

$$E^{(-1)}\Phi_1 = -v\Phi_2 \qquad E^{(-1)}\Phi_2 = \Phi_1 + \Lambda\Phi_2 \tag{5.1b}$$

$$\widetilde{P}_{2m} + \langle\Psi_1, \Phi_2\rangle = 0 . \tag{5.1c}$$

As we show in preceding section, (5.1) can be transformed into an integrable symplectic map.

The restricted flows of the KdV hierarchy is defined in [2, 3] as

$$\overline{\Psi}_{1x} = \overline{\Psi}_2 \qquad \overline{\Psi}_{2x} = (\overline{\Lambda} - u)\overline{\Psi}_1 \tag{5.2a}$$
$$\overline{\Phi}_{1x} = -(\overline{\Lambda} - u)\overline{\Phi}_2 \qquad \overline{\Phi}_{2x} = -\overline{\Phi}_1 \tag{5.2b}$$
$$\overline{P}_m + \langle\overline{\Psi}_1, \overline{\Phi}_2\rangle = 0 \tag{5.2c}$$

where the definition of $\overline{\Psi}_i$, $\overline{\Phi}_i$ and $\overline{\Lambda}$ is analogous to that for Ψ_i, Φ_i and Λ. It is shown in [2, 3] that (5.2) can be transformed into a finite-dimensional integrable Hamiltonian system (FDIHS).

As we show in appendix C, we have

Proposition 4. The continuous limit of (5.1) gives rise to (5.2).

Thus integrable symplectic map (5.1) provide numerical schemes for the finite-dimensional integrable Hamiltonian system (5.2).

For example, for $m = 1$ and $\widetilde{P}_2 = 1 - \frac{1}{v}$. Then equations (5.1$c$) and (5.1$b$) give

$$v = 1 + \langle\Psi_1, \widetilde{\Phi}_1\rangle . \tag{5.3}$$

By substituting (5.3) into (5.1a) and (5.1b) we obtain the following integrable symplectic map:

$$E\Psi_1 = \Psi_2 \qquad E\Psi_2 = -(1 + \langle\Psi_1, \widetilde{\Phi}_1\rangle)\Psi_1 + \Lambda\Psi_2 \tag{5.4a}$$
$$E\widetilde{\Phi}_1 = \frac{1}{1 + \langle\Psi_1, \widetilde{\Phi}_1\rangle}\Lambda\widetilde{\Phi}_1 + \widetilde{\Phi}_2 \qquad E\widetilde{\Phi}_2 = -\frac{1}{1 + \langle\Psi_1, \widetilde{\Phi}_1\rangle}\widetilde{\Phi}_1 . \tag{5.4b}$$

For $m = 1$ and $\overline{P}_1 = u$, equation (5.2c) gives

$$u = -\langle\overline{\Psi}_1, \overline{\Phi}_2\rangle . \tag{5.5}$$

Substituting (5.5) into (5.2a) and (5.2b) yields following system:

$$\overline{\Psi}_{1x} = \overline{\Psi}_2 \qquad \overline{\Psi}_{2x} = (\overline{\Lambda} + \langle\overline{\Psi}_1, \overline{\Phi}_2\rangle)\overline{\Psi}_1 \tag{5.6a}$$
$$\overline{\Phi}_{1x} = -(\overline{\Lambda} + \langle\overline{\Psi}_1, \overline{\Phi}_2\rangle)\overline{\Phi}_2 \qquad \overline{\Phi}_{2x} = -\overline{\Phi}_1 \tag{5.6b}$$

which is a FDIHS called as Garnier system. The continuous limit of ISM (5.4) gives, due to proposition 4, the FDIHS (5.6).

If we change the restriction (5.1c) and take

$$P_2 = 1 = -\langle\Psi_1, \Phi_2\rangle \tag{5.7}$$

then we find from (5.1b) that

$$v = \langle\Psi_1, \widetilde{\Phi}_1\rangle$$

and (5.1a), (5.1b) can be transformed into the following integrable symplectic map:

$$E\Psi_1 = \Psi_2 \qquad E\Psi_2 = -\langle\Psi_1, \widetilde{\Phi}_1\rangle\Psi_1 + \Lambda\Psi_2 \tag{5.8a}$$
$$E\widetilde{\Phi}_1 = \frac{1}{\langle\Psi_1, \widetilde{\Phi}_1\rangle}\Lambda\widetilde{\Phi}_1 + \widetilde{\Phi}_2 \qquad E\widetilde{\Phi}_2 = -\frac{1}{\langle\Psi_1, \widetilde{\Phi}_1\rangle}\widetilde{\Phi}_1 . \tag{5.8b}$$

Notice that $P_2 = \frac{1}{2}\overline{P}_0 = 1$, so the continuous limit of (5.7), due to (C.2) by taking $m = 0$, gives

$$\langle \overline{\Psi}_1, \overline{\Phi}_2 \rangle = 1. \tag{5.9a}$$

Thus the continuous limit of (5.8), according to (C.1) with $m = 0$, gives the well known Neumann system consisting of (5.9a) and of

$$\overline{\Psi}_{1x} = \overline{\Psi}_2 \qquad \overline{\Psi}_{2x} = (\overline{\Lambda} - \langle \overline{\Psi}_2, \overline{\Phi}_1 \rangle + \langle \overline{\Lambda\Psi}_1, \overline{\Phi}_2 \rangle)\overline{\Psi}_1 \tag{5.9b}$$

$$\overline{\Phi}_{1x} = -(\overline{\Lambda} - \langle \overline{\Psi}_2, \overline{\Phi}_1 \rangle + \langle \overline{\Lambda\Psi}_1, \overline{\Phi}_2 \rangle)\overline{\Phi}_2 \qquad \overline{\Phi}_{2x} = -\overline{\Phi}_1 . \tag{5.9c}$$

So the ISM (5.8) can be used for numerical schemes for calculation of the Neumann system (5.9).

Integrals of motion of (5.8) are also related to integrals of motion of (5.9). For instance, we have

$$F_1 = \overline{F}_1 h + O(h^2) \tag{5.10a}$$

$$F_2 + F_1 - 1 = \overline{F}_2 h^2 + O(h^3) \tag{5.10b}$$

where F_i and $\overline{F}_i$ are integrals of motion for (5.8) and (5.9), respectively, and given by [2, 3, 9, 10]

$$\begin{aligned}
F_1 &= \langle \Psi_1, \widetilde{\Phi}_1 \rangle + \langle \Psi_2, \widetilde{\Phi}_2 \rangle \\
F_2 &= \langle \Lambda^2\Psi_1, \widetilde{\Phi}_1 \rangle - \langle \Lambda^2\Psi_2, \widetilde{\Phi}_2 \rangle + \tfrac{1}{2}(\langle \Psi_1, \widetilde{\Phi}_1 \rangle - \langle \Psi_2, \widetilde{\Phi}_2 \rangle)^2 \\
&\qquad - 2\langle \Lambda\Psi_2, \widetilde{\Phi}_1 \rangle + 2\langle \Lambda\Psi_1, \widetilde{\Phi}_2 \rangle \langle \Psi_1, \widetilde{\Phi}_1 \rangle \\
\overline{F}_1 &= \langle \overline{\Psi}_1, \overline{\Phi}_1 \rangle + \langle \overline{\Psi}_2, \overline{\Phi}_2 \rangle \\
\overline{F}_2 &= -4\langle \overline{\Psi}_2, \overline{\Phi}_1 \rangle - 4\langle \overline{\Lambda\Psi}_1, \overline{\Phi}_2 \rangle + \tfrac{1}{2}(\langle \overline{\Psi}_1, \overline{\Phi}_1 \rangle - \langle \overline{\Psi}_2, \overline{\Phi}_2 \rangle)^2 .
\end{aligned}$$

In general, by using (C.1) and expanding in powers of h, we can find that the combination of integrals of motion $F_1, F_2, \ldots, F_k$ for (5.8) (or for (5.1)) goes to integral of motion $\overline{F}_k$ for (5.9) (or for (5.2)) in the continuous limit.

It would be interesting to compare the above method with other method for numerical scheme for calculation of FDIHS. We leave it for further numerical studies.

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Appendix A. Proof of proposition 1

It is easy to verify that for the Hamiltonian operators we get

$$J = -2h\partial - \left(\tfrac{1}{3}\partial^3 + 4u\partial + 2u_x\right)h^3 + O(h^5) \tag{A.1a}$$

$$J^{-1} = -\tfrac{1}{2}h^{-1}\partial^{-1} + \tfrac{1}{4}\partial^{-1}\left(\tfrac{1}{3}\partial^3 + 4u\partial + 2u_x\right)\partial^{-1}h + O(h^3) \tag{A.1b}$$

$$G = -8h\partial - \left(\tfrac{10}{3}\partial^3 + 24u\partial + 12u_x\right)h^3 + O(h^5) \tag{A.1c}$$

which gives the expansions

$$\tfrac{1}{4}J^{-1}G - 1 = B_0^{-1}B_1h^2 + \mathrm{O}(h^4) \tag{A.1d}$$

$$G - \lambda^2 J = -8(B_1 - \tilde{\lambda}B_0)h^3 + \mathrm{O}(h^5) \tag{A.1e}$$

which is just the formula (3.3).

Notice that

$$\tilde{P}_0 \equiv 4P_0 = \frac{2}{v} = 2 - 2uh^2 + \cdots = \overline{P}_0 + \mathrm{O}(h^2) \tag{A.2}$$

and

$$JP_0 = 0 \tag{A.3}$$

so by using (A.1*d*) we obtain

$$\left(\tfrac{1}{4}J^{-1}G - 1\right)\tilde{P}_0 = B_0^{-1}B_1\overline{P}_0h^2 + \mathrm{O}(h^3)\,.$$

It, together with (2.9*b*) and (2.29*a*), lead to

$$\tilde{P}_2 \equiv P_2 + \beta_{1,0}P_0 = \overline{P}_1h^2 + \mathrm{O}(h^3) \tag{A.4}$$

where $\beta_{1,0}$ is an undetermined constant. We now show (3.8) inductively. Using (A.1*d*) we have

$$\begin{aligned}\left(\tfrac{1}{4}J^{-1}G - 1\right)\sum_{i=0}^{k-1}\beta_{k-1,i}P_{2i} &= \tfrac{1}{4}\sum_{i=0}^{k-1}\beta_{k-1,i}P_{2(i+1)} + \gamma_kP_0 - \sum_{i=0}^{k-1}\beta_{k-1,i}P_{2i}\\ &= \tfrac{1}{4}\beta_{k-1,k-1}P_{2k} + \sum_{i=1}^{k-1}\left(\tfrac{1}{4}\beta_{k-1,i-1} - \beta_{k-1,i}\right)P_{2i} + \beta_{k,0}P_0\\ &= B_0^{-1}B_1\overline{P}_{k-1}h^{2k} + \mathrm{O}(h^{2k+1}) = \overline{P}_kh^{2k} + \mathrm{O}(h^{2k+1})\,.\end{aligned} \tag{A.5}$$

It gives (3.8*a*) and (3.8*b*) except the formula for $\beta_{k,0}$. It is known that the coefficient of u^k in $\overline{P}_k$ is $\frac{(2k-1)!!}{2^{k-1}k!}$. Observe that

$$P_0 = \frac{1}{2v} = \frac{1}{2(1+uh^2)} = \frac{1}{2}\sum_{i=0}^{\infty}(-u)^ih^{2i}$$

and the term u^kh^{2k} on the left-hand side of (3.8*a*) comes from P_0 only. Then by comparing the coefficients of u^kh^{2k} at both sides of (3.8*a*) we get immediately the formula for $\beta_{k,0}$ given by (3.8*b*). This completes the proof.

Appendix B. The involutivity of F_k

In order to prove involutivity of F_k, we consider equations following from (2.12), (2.15) and (2.23)

$$\Psi_{1,t_m} = \sum_{k=0}^{m} a_{2k}\Lambda^{2m-2k}\Psi_1 + \sum_{k=0}^{m-1} b_{2k+1}\Lambda^{2m-2k-1}\Psi_2 + b_{2m+1}\Psi_1 \tag{B.1a}$$

$$\Psi_{2,t_m} = \sum_{k=0}^{m-1} c_{2k+1}\Lambda^{2m-2k-1}\Psi_1 - \sum_{k=0}^{m} a_{2k}\Lambda^{2m-2k}\Psi_2 \tag{B.1b}$$

$$v_{t_m} = v(a_{2m}^{(-1)} - a_{2m}^{(1)}) \tag{B.1c}$$

$$\widetilde{\Phi}_{1,t_m} = -\sum_{k=0}^{m} a_{2k}\Lambda^{2m-2k}\widetilde{\Phi}_1 - \sum_{k=0}^{m-1} c_{2k+1}\Lambda^{2m-2k-1}\widetilde{\Phi}_2 - b_{2m+1}\widetilde{\Phi}_1 \tag{B.1d}$$

$$\widetilde{\Phi}_{2,t_m} = -\sum_{k=0}^{m-1} b_{2k+1}\Lambda^{2m-2k-1}\widetilde{\Phi}_1 + \sum_{k=0}^{m} a_{2k}\Lambda^{2m-2k}\widetilde{\Phi}_2 \tag{B.1e}$$

$$v_{t_m}^{(-1)} = v^{(-1)}(a_{2m}^{(-2)} - a_{2m}). \tag{B.1f}$$

By using (4.13) and (4.18), it is easy to verify by a straightforword calculation that equation (B.1) with a_k, b_k, c_k replaced by $\tilde{a}_k, \tilde{b}_k, \tilde{c}_k$ becomes a finite-dimensional Hamiltonian system (FDHS), i.e.

$$\Psi_{i,t_m} = \frac{\partial F_{m+2}}{\partial \widetilde{\Phi}_i} \qquad \widetilde{\Phi}_{i,t_m} = -\frac{\partial F_{m+2}}{\partial \Psi_i} \qquad i = 1, 2 \tag{B.2a}$$

$$q_{2N+1,t_m} = \frac{\partial F_{m+2}}{\partial p_{2N+1}} \qquad p_{2N+1,t_m} = -\frac{\partial F_{m+2}}{\partial q_{2N+1}}. \tag{B.2b}$$

Since $\widetilde{\Gamma}$ satisfies (2.12), according to (2.19) and (2.20), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t_m}(\tilde{a}^2 + \tilde{b}\tilde{c}) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t_m}F_k = 0 \qquad k, m = 0, 1, \ldots \tag{B.3}$$

which implies that the F_k are also integrals of motion for FDHS (B.2). The Poisson bracket for (B.2) are same as (4.4). So immediately from (B.2) and (B.3) we have

$$\{F_k, F_{m+2}\} = -\frac{\mathrm{d}}{\mathrm{d}t_m}F_k = 0 \qquad k, m = 0, 1, \ldots \tag{B.4}$$

which leads to (4.19) and means that integrals of motion F_k are in involution with respect to (4.4).

Appendix C. Proof of proposition 4

Let us, according to (3.1) by taking $\alpha = h^m$, (5.1) and (5.2), expand

$$E^{(k)}v(n) = 1 + u(x+kh)h^2 \qquad \Lambda = 2 + \overline{\Lambda}h^2 \tag{C.1a}$$

$$E^{(k)}\Psi_2(n) = h^m\overline{\Psi}_1(x+kh) \qquad E^{(k)}\Phi_2(n) = h^m\overline{\Phi}_2(x+kh) \tag{C.1b}$$

$$E^{(k)}\Psi_1(n) = h^m\left[\overline{\Psi}_1(x+kh) + \sum_{i=0}^{\infty}\frac{1}{(i+1)!}\partial^i\overline{\Psi}_2(x+kh)(-h)^{i+1}\right] \tag{C.1c}$$

$$E^{(k)}\Phi_1(n) = h^m\left[-\overline{\Phi}_2(x+kh) - \overline{\Lambda\Phi}_2(x+kh)h^2 - \sum_{i=0}^{\infty}\frac{1}{(i+1)!}\partial^i\overline{\Phi}_1(x+kh)(-h)^{i+1}\right]. \tag{C.1d}$$

Then

$$\langle\Psi_1, \Phi_2\rangle = \langle\Psi_2^{(-1)}, \Phi_2\rangle = h^{2m}\langle\overline{\Psi}_1, \overline{\Phi}_2\rangle + O(h^{2m+1}). \tag{C.2}$$

By using (3.8*a*) and (C.1), we get

$$E\Psi_1 - \Psi_2 = h^{m+1}(\overline{\Psi}_{1x} - \overline{\Psi}_2) + O(h^{m+2}) \tag{C.3a}$$

$$E\Psi_2 + v\Psi_1 - \Lambda\Psi_2 = h^{m+2}[\overline{\Psi}_{2x} - (\overline{\Lambda} - u)\overline{\Psi}_1] + O(h^{m+3}) \tag{C.3b}$$

$$E^{(-1)}\Phi_1 + v\Phi_2 = h^{m+2}[-\overline{\Phi}_{1x} - (\overline{\Lambda} - u)\overline{\Phi}_2] + O(h^{m+3}) \tag{C.3c}$$

$$E^{(-1)}\Phi_2 - \Phi_1 - \Lambda\Phi_2 = h^{m+1}(-\overline{\Phi}_{2x} - \overline{\Phi}_1) + O(h^{m+2}) \tag{C.3d}$$

$$\widetilde{P}_{2m} + \langle\Psi_1, \Phi_2\rangle = h^{2m}(\overline{P}_m + \langle\overline{\Psi}_1, \overline{\Phi}_2\rangle) + O(h^{2m+1}). \tag{C.3e}$$

This implies that the continuous limit of (5.1) gives rise to (5.2).

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